Phys 410
Fall 2015
Lecture #13 Summary
13 October, 2015

Constrained systems are common in physics, and their dynamics can be advantageously solved by the Lagrangian method. Examples include the pendulum, the Atwood machine, a rigid body, gas particles trapped in a box, a rolling object, and a bead on a wire. We considered the pendulum problem in detail. The constraint is that the length of the pendulum ℓ is fixed, so that the x- and y-coordinates of the bob are not independent, but constrained so that $\ell = \sqrt{x^2 + y^2}$. We can elegantly incorporate this constraint by adopting a new independent variable to describe the position of the bob, namely the angle that the pendulum makes with the vertical, ϕ. In terms of this generalized coordinate, the Lagrangian becomes $\mathcal{L}(\phi, \dot{\phi}) = \frac{m}{2}\ell^2\dot{\phi}^2 - mg\ell(1 - \cos\phi)$. Lagrange's equation gives $-mg\ell\sin\phi = m\ell^2\ddot{\phi}$, which relates the torque due to gravity on the bob to the time rate of change of the angular momentum of the bob, or the moment of inertia ($m\ell^2$) times the angular acceleration ($\ddot{\phi}$). Note that the force of constraint (namely the tension in the rod supporting the bob) never played a role in the analysis (whereas it plays an important role in the traditional Newtonian approach). Once the appropriate generalized coordinate is identified, the associated constraining force disappears from the discussion!

Generalized coordinates and constrained systems are important for Lagrangian dynamics. Consider a system consisting of N particles, with positions $\vec{r_\alpha}$, with $\alpha = 1, \dots N$. The parameters $q_1, q_2, \dots q_n$ are a set of generalized coordinates if each position $\vec{r_\alpha}$ can be expressed as a function of $q_1, q_2, \dots q_n$, and possibly time t as, $\vec{r_\alpha} = \vec{r_\alpha}(q_1, q_2, \dots q_n, t)$for $\alpha = 1, \dots N$, and the inverse $q_i = q_i(\vec{r_1}, \vec{r_2}, \dots \vec{r_N}, t)$ for $i = 1, 2, \dots n$ can also be written. For particles in three dimensions, $n \leq 3N$. If $n < 3N$, then the system is said to be **constrained**. The number of degrees of freedom of a system is the number of coordinates that can be independently varied in a small displacement. The simple pendulum is constrained and has one degree of freedom. The double pendulum is constrained and has two degrees of freedom. One can show (the proof is in Taylor, section 7.4) that constrained systems with holonomic constraints obey the Lagrange equations when their Lagrangian is written in terms of the generalized coordinates of the system. Holonomic constraints are those which impose relations between only the coordinates of the system. Non-holonomic constraints cannot be reduced to relations only between the coordinates. For example consider a rolling wheel on a fixed surface – the rolling constraint says that the velocity at the point of contact is zero. This is a condition on a quantity other than the coordinates of the particles.

We discussed several examples of constrained systems by the Lagrangian method. The key step is to identify the number of degrees of freedom in the problem and find the most efficient set of generalized coordinates. Examining the constraints in the system is often a good way to identify the appropriate generalized coordinates. Writing down the kinetic and potential energies in terms of these generalized coordinates is often facilitated by using Cartesian or cylindrical or spherical coordinates, and then converting completely to the generalized coordinates.

We did the example of the Atwood machine for a frictionless and inertia-less pulley supporting two different masses. The masses can each move in one dimension (which we called x and y), and their motion is constrained because they are on either end of a string of fixed length. The constraint is that the string length is $\ell = x + y + \pi R$, where R is the radius of the pulley. With this constraint incorporated, the Lagrangian can be written as $\mathcal{L}(x, \dot{x}) = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx + const.$ Note that the constant plays no role in the dynamics since it disappears when any of the derivatives $(\frac{\partial \mathcal{L}}{\partial x}, \frac{\partial \mathcal{L}}{\partial \dot{x}}, \frac{d}{dt})$ are taken. The resulting equation of motion is $\ddot{x} = g\frac{m_1 - m_2}{m_1 + m_2}$. Again note that the constraining force (the tension in the string) was never mentioned or considered in the process. The tension is essential to the traditional Newton's second law approach to solving this problem.

We then looked at the pendulum whose point of suspension is forced to rotate on a circle of radius R at a fixed angular velocity ω. The key step is to write down the (x, y) coordinates of the bob in terms of a minimum number of parameters and generalized coordinates. We did this by describing the location of the bob starting from the center axis of the circle (chosen to be the origin) and describing the location of the point of suspension, and then adding the vector position of the bob relative to the point of suspension. The location of the particle is specified by a single variable, φ, which describes the deviation of the bob from the vertical. This location (x, y) was then differentiated with respect to time to get the vector velocity (in terms of φ, $\dot{\varphi}$, and time t), and the kinetic energy was constructed from that. The potential energy is entirely due to gravity, so the Lagrangian can be constructed. The Euler-Lagrange equation yields the equation of motion for the single generalized coordinate φ. The resulting motion can be quite complicated, and we will study the development of chaos in the driven damped pendulum later in the semester.